Derivation of the flux of substance X towards roots using the single-root concept

Tobias Klöffel

Department of Plant and Environmental Sciences, University of Copenhagen

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We start with Equation (7.24) in the documentation, which formulates the advection-dispersion equation for a cylindrical/radial geometry:

$$\frac{\delta(\theta C_x)}{\delta t} = \frac{1}{r}\frac{\delta}{\delta r}\left(rD_x^*\frac{\delta N_x}{\delta r} + rv_wC_x\right) \tag{1}$$

where θ is the soil water content [m^3 m^{-3}], N_x is the concentration of X in the bulk soil [kg m^{-3}], C_x is the concentration of X in the soil solution [kg m^{-3}], v_w is the water flux density [m^3 m^{-2} s^{-1}], r is the radial distance from the center of the root [m], and D_x^* is the dispersion coefficient of substance X [m^2 s^{-1}].

We can define a so-called buffer power with respect to substance X, referred to as b_x, as

$$b_x = \frac{N_x}{C_x} \tag{2}$$

and re-arranging gives

$$N_x = b_xC_x \tag{3}$$

b_x can therefore be used to convert the concentration of substance X with respect to the *soil solution* to the concentration of substance X with respect to the *bulk soil*. Note that b_x is basically equivalent to θ for nitrate, while, for ammonium, sorption is also taken into account (see Equations 7.39 and 7.40 in the documentation).

The flux of substance X towards the root in a cylindrical domain can be split up between a convection part, given as

$$I_{x,c} = q_wC_x$$

where $q_w = 2\pi rv_w$ [m^3 m^{-1} s^{-1}], and a diffusion part, given as

$$I_{x,d} = 2\pi rD_x^*b_x\frac{\delta C}{\delta r} \tag{4}$$

To make the following equations more compact, we define $D_x = D_x^*b_x$. Finally, summing up both parts gives the total flux of substance X towards the root, which is equal to Equation (7.26) in the documentation:

$$I_x = 2\pi rD_x\frac{\delta C}{\delta r} + q_wC_x \tag{5}$$

To determine C_x at a specific radius r and arrive at Equation (7.28) in the documentation, we need to solve Eq. [5]. For this, we first define

$$\alpha = \frac{q_w}{2\pi D_x}$$

which we will use for substitution in the following equations. Then we re-arrange Eq. [5] such that

$$\frac{\delta C_x}{\delta r} + \frac{\alpha}{r}C_x = \frac{I_x}{2\pi rD_x}. \tag{6}$$

This is a first-order linear ordinary differential equation of the form

$$\frac{\delta y}{\delta x} + P(x)y = Q(x).$$

Thus, to solve the equation, we can multiply both sides of Eq. [6] by $\mu(x)$, which we define as

$$\mu(x) = e^{\int P(x)dx} = e^{\alpha \ln(r)} = r^{\alpha}$$

Now we multiply both sides of Eq. [6] with r^{α}, where the left side becomes

$$r^{\alpha}\frac{\delta C_x}{\delta r} + r^{\alpha}\frac{\alpha}{r}C_x = r^{\alpha}\frac{\delta C_x}{\delta r} + \alpha r^{\alpha-1}C_x = \frac{\delta}{dr}r^{\alpha}C_x$$

and the last equality holds due to the product rule of differentiation. Combining both sides yields

$$\frac{\delta}{dr}r^{\alpha}C_x = r^{\alpha-1}\frac{I_x}{2\pi D_x}. \tag{7}$$

Once we integrate Eq. [7], we will realise that the equation becomes undefined for $\alpha = 0$. Thus, we need to treat both cases. Setting $\alpha = 0$ in Eq. [7] and integrating, gives

$$C_x = \frac{I_x}{2\pi D_x}\int r^{-1}dx = \frac{I_x}{2\pi D_x}\ln(r) + B$$

where B is a constant. For the case where $\alpha \neq 0$, integration of Eq. [7] gives

$$r^{\alpha}C_x = \frac{I_x}{2\pi D_x}\int r^{\alpha-1}dx = \frac{I_x}{2\pi D_x}\frac{r^{\alpha}}{\alpha} + B.$$

If we divide by r^{α}, we get

$$C_x = \frac{I_x}{2\pi D_x \alpha} + Br^{-\alpha} = \frac{I_x}{q_w} + Br^{-\alpha}.$$

This gives us the two solutions, which equal Equation 7.28 in the documentation:

$$C_x = \begin{cases} \frac{I_x}{2\pi D_x}\ln(r) + B, & \alpha = 0. \\ \frac{I_x}{q_w} + Br^{-\alpha}, & \alpha \neq 0. \end{cases} \tag{8}$$

Determine the integration constants B

In the next step, we will derive the integration constants B in Eq. [8]. For this we set $C_x = C_{x,0}$ and $r = r_r$, where $C_{x,0}$ is the concentration of substance X at the root surface and r_r is the root radius. We start with the case where $\alpha = 0$, followed by $\alpha \neq 0$.

$\underline{\alpha = 0}$

Re-arranging Eq. [8] for $\alpha = 0$ yields

$$B = C_{x,0} - \frac{I_x}{2\pi D_x}\ln(r_r).$$

Thus, Eq. [8] becomes

$$C_x = \frac{I_x}{2\pi D_x}\ln(r) + C_{x,0} - \frac{I_x}{2\pi D_x}\ln(r_r) = C_{x,0} + \frac{I_x}{2\pi D_x}\ln\left(\frac{r}{r_r}\right). \tag{9}$$

$\underline{\alpha \neq 0}$

Re-arranging Eq. [8] for $\alpha \neq 0$ yields

$$B = \frac{C_{x,0}}{r_r^{-\alpha}} - \frac{I_x}{q_w r_r^{-\alpha}} = C_{x,0}r_r^{\alpha} - \frac{I_x}{q_w}r_r^{\alpha} = \frac{1}{r_r^{-\alpha}}\left(C_{x,0} - \frac{I_x}{q_w}\right).$$

Thus, Eq. [8] becomes

$$\begin{aligned} C_x = \frac{I_x}{q_w} + \left[\left(C_{x,0} - \frac{I_x}{q_w}\right)\frac{1}{r_r^{-\alpha}}\right]r^{-\alpha} &= \frac{I_x}{q_w} + C_{x,0}\frac{r^{-\alpha}}{r_r^{-\alpha}} - \frac{I_x}{q_w}\frac{r^{-\alpha}}{r_r^{-\alpha}} \\ &= \frac{I_x}{q_w} + \left[C_{x,0} - \frac{I_x}{q_w}\right]\left[\frac{r}{r_r}\right]^{-\alpha}. \end{aligned} \tag{10}$$

Now our two solutions equal Equation 7.30 in the documentation.

Determine the average concentration of substance X in the cylindrical domain ($\overline{C_x}$)

We will now derive the average concentration of substance X ($\overline{C_x}$) over the cylindrical domain. For this we integrate Eq. [9] and [10] from the root radius (r_r) to the outer part of the cylinder (r_c), and divide the result by the area of the domain. The domain has a doughnut shape for which the area, A, can be calculated as

$$A = \pi r_c^2 - \pi r_r^2 = \pi\left(r_c^2 - r_r^2\right)$$

We start by formulating the equation, which is equal to Equation (7.31) in the documentation

$$\overline{C_x} = \frac{1}{A}\int_{r_r}^{r_c} C_x dA = \frac{1}{\pi\left(r_c^2 - r_r^2\right)}\int_{r_r}^{r_c} C_x 2\pi r dr = \frac{2\pi}{\pi\left(r_c^2 - r_r^2\right)}\int_{r_r}^{r_c} rC_x dr \tag{11}$$

Now we will solve Eq. [11] by inserting Eq. [9] and [10] for the cases $\alpha = 0$ and $\alpha \neq 0$ respectively. For $\alpha \neq 0$, we will realise that the equation will be undefined for $\alpha = 2$. Thus, we need to treat three cases here, namely, $\alpha = 0$, $\alpha \neq 0 \wedge \alpha \neq 2$, and $\alpha = 2$. We start with $\alpha = 0$:

$\underline{\alpha = 0}$

$$
\begin{aligned}
\overline{C_x} &= \frac{2\cancel{\pi}}{\cancel{\pi}\left(r_c^2 - r_r^2\right)} \int_{r_r}^{r_c} C_x r dr = \frac{2}{\left(r_c^2 - r_r^2\right)} \int_{r_r}^{r_c} \left[C_{x,0} + \frac{I_x}{2\pi D_x} \ln\left(\frac{r}{r_r}\right) \right] r dr \\
&= \frac{2}{\left(r_c^2 - r_r^2\right)} \left[C_{x,0} \int_{r_r}^{r_c} r dr + \frac{I_x}{2\pi D_x} \int_{r_r}^{r_c} r \ln\left(\frac{r}{r_r}\right) dr \right]
\end{aligned} \tag{12}
$$

We will split up the expression in brackets and deal with the two integrals separately:

Left integral

$$
C_{x,0} \int_{r_r}^{r_c} r dr = C_{x,0} \left[\frac{r_c^2}{2} - \frac{r_r^2}{2} \right] = \frac{C_{x,0}}{2} \left(r_c^2 - r_r^2\right)
$$

Right integral

$$
\begin{aligned}
\frac{I_x}{2\pi D_x} \int_{r_r}^{r_c} r \ln\left(\frac{r}{r_r}\right) dr &= \frac{I_x}{2\pi D_x} \int_{r_r}^{r_c} r \left[\ln(r) - \ln(r_r)\right] dr = \frac{I_x}{2\pi D_x} \int_{r_r}^{r_c} r \ln(r) dr - \ln(r_r) \int_{r_r}^{r_c} r dr \\
&= \frac{I_x}{2\pi D_x} \left[\frac{1}{2} r^2 \ln(r) - \frac{1}{2} \int_{r_r}^{r_c} \frac{1}{\cancel{r}} r^{\cancel{2}} dr - \ln(r_r) \frac{1}{2} \left(r_c^2 - r_r^2\right) \right] \\
&= \frac{I_x}{2\pi D_x} \left[\frac{1}{2} r_c^2 \ln(r_c) - \frac{1}{4} r_c^2 - \frac{1}{2} r_r^2 \ln(r_r) + \frac{1}{4} r_r^2 - \ln(r_r) \frac{1}{2} \left(r_c^2 - r_r^2\right) \right] \\
&= \frac{I_x}{2\pi D_x} \left[\frac{1}{2} \left(r_c^2 \ln(r_c) - r_r^2 \ln(r_r) - \frac{1}{2} \left(r_c^2 - r_r^2\right) - \ln(r_r) \left(r_c^2 - r_r^2\right) \right) \right]
\end{aligned}
$$

Inserting both solutions back into Eq. [12] gives

$$
\begin{aligned}
\overline{C_x} &= \frac{2}{\left(r_c^2 - r_r^2\right)} \left[\frac{C_{x,0}}{2} \left(r_c^2 - r_r^2\right) + \frac{I_x}{2\pi D_x} \left[\frac{1}{2} \left(r_c^2 \ln(r_c) - r_r^2 \ln(r_r) - \frac{1}{2} \left(r_c^2 - r_r^2\right) - \ln(r_r) \left(r_c^2 - r_r^2\right) \right) \right] \right] \\
&= C_{x,0} + \frac{I_x}{2\pi D_x} \left(\frac{r_c^2 \ln(r_c) - r_r^2 \ln(r_r)}{\left(r_c^2 - r_r^2\right)} - \ln(r_r) - \frac{1}{2} \right).
\end{aligned} \tag{13}
$$

Now we can use the substitution

$$
\beta = \frac{1}{r_r \sqrt{\pi L}} = \frac{r_c}{r_r} \qquad \Rightarrow \qquad r_c = \beta r_r
$$

where L is the root length density [m m^{-3}], and apply it to Eq. [13]:

$$
\begin{aligned}
\overline{C_x} &= C_{x,0} + \frac{I_x}{2\pi D_x} \left(\frac{\beta^2 r_r^2 \left(\ln(\beta) + \ln(r_r)\right) - r_r^2 \ln(r_r)}{r_r^2 \left(\beta^2 - 1\right)} - \ln(r_r) - \frac{1}{2} \right) \\
&= C_{x,0} + \frac{I_x}{2\pi D_x} \left(\frac{\cancel{r_r^2} \left(\beta^2 \left(\ln(\beta) + \ln(r_r)\right) - \ln(r_r)\right)}{\cancel{r_r^2} \left(\beta^2 - 1\right)} - \ln(r_r) - \frac{1}{2} \right) \\
&= C_{x,0} + \frac{I_x}{2\pi D_x} \left(\frac{\beta^2 \ln(\beta)}{\left(\beta^2 - 1\right)} + \frac{\cancel{\left(\beta^2 - 1\right)} \ln(r_r)}{\cancel{\left(\beta^2 - 1\right)}} - \ln(r_r) - \frac{1}{2} \right) \\
&= C_{x,0} + \frac{I_x}{2\pi D_x} \left(\frac{\beta^2 \ln(\beta)}{\left(\beta^2 - 1\right)} - \frac{1}{2} \right)
\end{aligned} \tag{14}
$$

$\underline{\alpha \neq 0 \land \alpha \neq 2}$

$$
\begin{aligned}
\overline{C_x} &= \frac{2\cancel{\pi}}{\cancel{\pi}\left(r_c^2 - r_r^2\right)} \int_{r_r}^{r_c} C_x r dr = \frac{2}{\left(r_c^2 - r_r^2\right)} \int_{r_r}^{r_c} \left[\frac{I_x}{q_w} + \left[C_{x,0} - \frac{I_x}{q_w}\right]\left[\frac{r}{r_r}\right]^{-\alpha}\right] r dr \\
&= \frac{2}{\left(r_c^2 - r_r^2\right)} \left(\int_{r_r}^{r_c} \frac{I_x}{q_w} r dr + \int_{r_r}^{r_c} \left[C_{x,0} - \frac{I_x}{q_w}\right]\left[\frac{r}{r_r}\right]^{-\alpha} r dr\right) \\
&= \frac{2}{\left(r_c^2 - r_r^2\right)} \left(\frac{I_x}{q_w} \int_{r_r}^{r_c} r dr + \left[C_{x,0} - \frac{I_x}{q_w}\right] \int_{r_r}^{r_c} \left[\frac{r}{r_r}\right]^{-\alpha} r dr\right)
\end{aligned} \tag{15}
$$

We will again split up the expression in brackets and deal with the two integrals separately:

Left integral

$$
\frac{I_x}{q_w} \int_{r_r}^{r_c} r dr = \frac{I_x}{q_w}\left[\frac{r_c^2}{2} - \frac{r_r^2}{2}\right] = \frac{I_x}{2q_w}\left(r_c^2 - r_r^2\right)
$$

Right integral

$$
\begin{aligned}
\left[C_{x,0} - \frac{I_x}{q_w}\right] \int_{r_r}^{r_c} \left[\frac{r}{r_r}\right]^{-\alpha} r dr &= \left[C_{x,0} - \frac{I_x}{q_w}\right] r_r^\alpha \int_{r_r}^{r_c} r^{1-\alpha} dr = \left[C_{x,0} - \frac{I_x}{q_w}\right] r_r^\alpha \left[\frac{r_c^{2-\alpha}}{2-\alpha} - \frac{r_r^{2-\alpha}}{2-\alpha}\right] \\
&= \left[C_{x,0} - \frac{I_x}{q_w}\right] r_r^\alpha \frac{r_c^{2-\alpha} - r_r^{2-\alpha}}{2-\alpha} = \left[C_{x,0} - \frac{I_x}{q_w}\right] \frac{r_r^\alpha}{2-\alpha}\left(r_c^{2-\alpha} - r_r^{2-\alpha}\right)
\end{aligned}
$$

The solution of the right integral illustrates that, if $\alpha = 2$, the equation becomes undefined.

Inserting the solutions of both integrals back into Eq. [15] gives

$$
\begin{aligned}
\overline{C_x} &= \frac{2}{\left(r_c^2 - r_r^2\right)} \left[\frac{I_x}{2q_w}\left(r_c^2 - r_r^2\right) + \left[C_{x,0} - \frac{I_x}{q_w}\right] \frac{r_r^\alpha}{2-\alpha}\left(r_c^{2-\alpha} - r_r^{2-\alpha}\right)\right] \\
&= \frac{I_x}{q_w} + \frac{2}{\left(r_c^2 - r_r^2\right)} \left[C_{x,0} - \frac{I_x}{q_w}\right] \left[\frac{r_r^\alpha}{2-\alpha}\left(r_c^{2-\alpha} - r_r^{2-\alpha}\right)\right] \\
&= \frac{I_x}{q_w} + \frac{2}{\left(r_c^2 - r_r^2\right)} \left[C_{x,0} - \frac{I_x}{q_w}\right] \left[\frac{r_r^\alpha r_c^{2-\alpha} - r_r^2}{2-\alpha}\right].
\end{aligned} \tag{16}
$$

Now we can again use the substitution

$$
\beta = \frac{1}{r_r\sqrt{\pi L}} = \frac{r_c}{r_r} \qquad \Rightarrow \qquad r_c = \beta r_r
$$

and apply it to Eq. [16], which yields

$$
\begin{aligned}
\overline{C_x} &= \frac{I_x}{q_w} + \left[C_{x,0} - \frac{I_x}{q_w}\right] \frac{2}{\left(\beta^2 r_r^2 - r_r^2\right)} \left[\frac{r_r^\alpha \left(\beta r_r\right)^{2-\alpha} - r_r^2}{2-\alpha}\right] \\
&= \frac{I_x}{q_w} + \left[C_{x,0} - \frac{I_x}{q_w}\right] \frac{2}{\cancel{r_r^2}\left(\beta^2 - 1\right)} \left[\frac{\cancel{r_r^2}\left(\beta^{2-\alpha} - 1\right)}{2-\alpha}\right] \\
&= \frac{I_x}{q_w} + \left[C_{x,0} - \frac{I_x}{q_w}\right] \frac{\left(\beta^{2-\alpha} - 1\right)}{\left(\beta^2 - 1\right)\left(1 - \frac{\alpha}{2}\right)}
\end{aligned} \tag{17}
$$

where for the last equality we used $2 - \alpha = 2\left(1 - \frac{\alpha}{2}\right)$.

$\underline{\alpha = 2}$

For the case $\alpha = 2$, we start out with Eq. [15]:

$$\overline{C_x} = \frac{2}{(r_c^2 - r_r^2)} \left(\frac{I_x}{q_w} \int_{r_r}^{r_c} r dr + \left[C_{x,0} - \frac{I_x}{q_w} \right] \int_{r_r}^{r_c} \left[\frac{r}{r_r} \right]^{-\alpha} r dr \right)$$

While the solution of the left integral of the expression in brackets can be adopted from above, we need to look at the right integral. If we set $\alpha = 2$, this becomes

$$\begin{aligned} \left[C_{x,0} - \frac{I_x}{q_w} \right] \int_{r_r}^{r_c} \left[\frac{r}{r_r} \right]^{-2} r dr &= \left[C_{x,0} - \frac{I_x}{q_w} \right] \int_{r_r}^{r_c} \left[\frac{r_r^2}{r^{\not 2}} \right] \not r dr \\ = \left[C_{x,0} - \frac{I_x}{q_w} \right] r_r^2 \int_{r_r}^{r_c} \frac{1}{r} dr &= \left[C_{x,0} - \frac{I_x}{q_w} \right] r_r^2 \left(\ln r_c - \ln r_r \right) \\ = \left[C_{x,0} - \frac{I_x}{q_w} \right] r_r^2 \left(\ln (\beta r_r) - \ln r_r \right) &= \left[C_{x,0} - \frac{I_x}{q_w} \right] r_r^2 \left(\ln \frac{\beta \not r_r}{\not r_r} \right) \\ &= \left[C_{x,0} - \frac{I_x}{q_w} \right] r_r^2 \ln \beta \end{aligned}$$

where we again applied the substitution $r_c = \beta r_r$. Inserting the solutions of both integrals back into Eq. [15], we obtain

$$\begin{aligned} \overline{C_x} &= \frac{2}{(r_c^2 - r_r^2)} \left[\frac{I_x}{2q_w} \left(r_c^2 - r_r^2 \right) + \left[C_{x,0} - \frac{I_x}{q_w} \right] r_r^2 \ln \beta \right] \\ = \frac{I_x}{q_w} + \frac{2}{(r_c^2 - r_r^2)} \left[\left(C_{x,0} - \frac{I_x}{q_w} \right) r_r^2 \ln \beta \right] &= \frac{I_x}{q_w} + \left(C_{x,0} - \frac{I_x}{q_w} \right) \frac{2}{(r_c^2 - r_r^2)} r_r^2 \ln \beta \\ = \frac{I_x}{q_w} + \left(C_{x,0} - \frac{I_x}{q_w} \right) \frac{2}{\not r_r^{\not 2} (\beta^2 - 1)} \not r_r^{\not 2} \ln \beta &= \frac{I_x}{q_w} + \left(C_{x,0} - \frac{I_x}{q_w} \right) \frac{\ln \beta^2}{(\beta^2 - 1)}. \end{aligned} \tag{18}$$

This gives us the three solutions, which equal Equation (7.32) in the documentation:

$$\overline{C_x} = \begin{cases} C_{x,0} + \dfrac{I_x}{2\pi D_x} \left(\dfrac{\beta^2 \ln(\beta)}{(\beta^2 - 1)} - \dfrac{1}{2} \right), & \alpha = 0. \\ \dfrac{I_x}{q_w} + \left[C_{x,0} - \dfrac{I_x}{q_w} \right] \dfrac{\left(\beta^{2-\alpha} - 1 \right)}{(\beta^2 - 1) \left(1 - \dfrac{\alpha}{2} \right)}, & \alpha \neq 0 \wedge \alpha \neq 2. \\ \dfrac{I_x}{q_w} + \left(C_{x,0} - \dfrac{I_x}{q_w} \right) \dfrac{\ln \beta^2}{(\beta^2 - 1)}, & \alpha = 2. \end{cases} \tag{19}$$

Determine I_x

As a last step, we need to solve the equations in Eq. [19] for I_x, which is mainly done by re-arranging. We again go through the three case, starting with $\alpha = 0$.

$\underline{\alpha = 0}$

$$
\begin{aligned}
\overline{C_x} &= C_{x,0} + \frac{I_x}{2\pi D_x}\left(\frac{\beta^2 \ln(\beta)}{(\beta^2-1)} - \frac{1}{2}\right) \\
\iff \left(\overline{C_x} - C_{x,0}\right) &= \frac{I_x}{2\pi D_x}\left(\frac{\beta^2 \ln(\beta)}{(\beta^2-1)} - \frac{1}{2}\right) \\
\iff I_x &= \left(\overline{C_x} - C_{x,0}\right) 2\pi D_x \left[\frac{\beta^2 \ln(\beta)}{(\beta^2-1)} - \frac{1}{2}\right]^{-1}
\end{aligned}
\tag{20}
$$

$\underline{\alpha \neq 0 \wedge \alpha \neq 2}$

Here we first slightly re-write Eq. [19]:

$$
\begin{aligned}
\overline{C_x} &= \frac{I_x}{q_w} + \left[C_{x,0} - \frac{I_x}{q_w}\right]\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)} \\
&= \frac{I_x}{q_w} + C_{x,0}\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)} - \frac{I_x}{q_w}\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)} \\
&= \frac{I_x}{q_w}\left(1 - \frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}\right) + C_{x,0}\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}
\end{aligned}
$$

Now we can solve for I_x:

$$
\begin{aligned}
\overline{C_x} &= \frac{I_x}{q_w}\left(1 - \frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}\right) + C_{x,0}\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)} \\
\iff \overline{C_x} - C_{x,0}\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)} &= \frac{I_x}{q_w}\left(1 - \frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}\right) \\
\iff \frac{I_x}{q_w} &= \frac{\overline{C_x} - C_{x,0}\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}}{1 - \frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}} \\
\iff I_x &= q_w\frac{\overline{C_x} - C_{x,0}\frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}}{1 - \frac{\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}}
\end{aligned}
$$

We can simplify this equation by multiplying $\overline{C_x}$ in the numerator and 1 in the denominator by the term

$$
\frac{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}
$$

which yields

$$
\begin{aligned}
I_x &= q_w\frac{\overline{C_x}\frac{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}{\cancel{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}} - C_{x,0}\frac{\left(\beta^{2-\alpha}-1\right)}{\cancel{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}}}{\frac{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}{\cancel{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}} - \frac{\left(\beta^{2-\alpha}-1\right)}{\cancel{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right)}}} \\
&= q_w\frac{\overline{C_x}\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right) - C_{x,0}\left(\beta^{2-\alpha}-1\right)}{\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right) - \left(\beta^{2-\alpha}-1\right)}.
\end{aligned}
\tag{21}
$$

$\underline{\alpha = 2}$

Also here we first slightly re-write Eq. [19]:

$$
\begin{aligned}
\overline{C_x} = \frac{I_x}{q_w} + \left(C_{x,0} - \frac{I_x}{q_w}\right)\frac{\ln\beta^2}{(\beta^2-1)} &= \frac{I_x}{q_w} + C_{x,0}\frac{\ln\beta^2}{(\beta^2-1)} - \frac{I_x}{q_w}\frac{\ln\beta^2}{(\beta^2-1)} \\
&= \frac{I_x}{q_w}\left(1 - \frac{\ln\beta^2}{(\beta^2-1)}\right) + C_{x,0}\frac{\ln\beta^2}{(\beta^2-1)}
\end{aligned}
\tag{22}
$$

Now we solve for I_x:

$$
\begin{aligned}
\overline{C_x} &= \frac{I_x}{q_w}\left(1 - \frac{\ln\beta^2}{(\beta^2-1)}\right) + C_{x,0}\frac{\ln\beta^2}{(\beta^2-1)} \\
\iff \overline{C_x} - C_{x,0}\frac{\ln\beta^2}{(\beta^2-1)} &= \frac{I_x}{q_w}\left(1 - \frac{\ln\beta^2}{(\beta^2-1)}\right) \\
\iff I_x &= q_w\frac{\overline{C_x} - C_{x,0}\frac{\ln\beta^2}{(\beta^2-1)}}{1 - \frac{\ln\beta^2}{(\beta^2-1)}}
\end{aligned}
\tag{23}
$$

This time we can simplify the equation by multiplying $\overline{C_x}$ in the numerator and 1 in the denominator by the term

$$
\frac{(\beta^2-1)}{(\beta^2-1)}
$$

which yields

$$
\begin{aligned}
I_x &= q_w\frac{\overline{C_x}\frac{(\beta^2-1)}{\cancel{(\beta^2-1)}} - C_{x,0}\frac{\ln\beta^2}{\cancel{(\beta^2-1)}}}{\frac{(\beta^2-1)}{\cancel{(\beta^2-1)}} - \frac{\ln\beta^2}{\cancel{(\beta^2-1)}}} \\
&= q_w\frac{\overline{C_x}\left(\beta^2-1\right) - C_{x,0}\ln\beta^2}{(\beta^2-1) - \ln\beta^2}.
\end{aligned}
\tag{24}
$$

This gives us the three solutions for I_x, which equal Equation (7.33) in the documentation:

$$
I_x = \begin{cases}
\left(\overline{C_x} - C_{x,0}\right)2\pi D_x\left[\frac{\beta^2\ln(\beta)}{(\beta^2-1)} - \frac{1}{2}\right]^{-1}, & \alpha = 0. \\
q_w\frac{\overline{C_x}\left(\beta^2-1\right)\left(1-\frac{\alpha}{2}\right) - C_{x,0}\left(\beta^{2-\alpha}-1\right)}{(\beta^2-1)\left(1-\frac{\alpha}{2}\right) - (\beta^{2-\alpha}-1)}, & \alpha \neq 0 \wedge \alpha \neq 2. \\
q_w\frac{\overline{C_x}\left(\beta^2-1\right) - C_{x,0}\ln\beta^2}{(\beta^2-1) - \ln\beta^2}, & \alpha = 2.
\end{cases}
\tag{25}
$$