Appendix 10.2
Estimating the root density from root dry matter and size (option *GPD1* and *GPD2* of *rootdens*).

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This Appendix is a copy of Annex 3.4, Estimating root density in Daisy, from the deliverable D3.2, Final version and documented models available, of the Safir project, submitted 30.04 2009. The text is slightly edited by Merete Styczen in this version.

Abstract

In this paper, we (Abrahamsen, Mollerup and Hansen) extend an empirical root density distribution function (Gerwitz and Page, 1974) based on densely populated homogeneous fields to row crops. The row crops are modeled as having a uniform density in the direction parallel to the rows, but variable in the direction perpendicular to the row. In each case (homogenous fields and row crops), the method to find the distribution parameters from the root dry matter and the size of the root zone is described.

1 Densely populated (homogenous) fields

In accordance with Gerwitz and Page (1974), the root density distribution, L_z, for a crop can be described as:

$$L_z = L_0 \cdot e^{-a \cdot z} \tag{10.2.1}$$

where L_0 is the root density at the soil surface, a is a distribution parameter and z is the depth below soil surface.

We assume here that the density is uniformly distributed on the horizontal plane, an assumption that fails with e.g. row crops. The parameters a and L_0 will both vary with time. For a production oriented simulation model like Daisy (Hansen et al., 1991; Abrahamsen and Hansen, 2000), it can be more convenient to specify the density in terms of accumulated root matter, M_r, and the total root zone depth, d_c, as described in Hansen et al. (1990) and repeated below.

We define the root depth at the lowest depth where the root density is above or at a specified threshold, L_m. By inserting this in eq. (10.3.1), we get:

$$L_m = L_{d_c} = L_0 \cdot e^{-a d_c} \tag{10.2.2}$$

We convert the root mass to root length, l_r, by assuming the specific root length, S_r, is a known constant (rather than varying with depth):

$$l_r = S_r M_r \tag{10.2.3}$$

The total root length is also the integral of the root density over the profile:

$$l_r = \int_0^\infty L_z dz = \int_0^\infty L_0 e^{-az} dz = \frac{L_0}{a} \tag{10.2.4}$$

From eq. (10.2.4), L_0 can be isolated, and inserted into eq. (10.2.2):

$$L_m = l_r \cdot a \cdot e^{-ad_c} \tag{10.2.5}$$

If we now define $W = -a \cdot d_c$, substitute the value into eq. (10.2.5), and isolate the known values on the right side, the result is:

$$W \cdot e^W = -L_m \frac{d_c}{l_r} \tag{10.2.6}$$

The solution to this equation with regard to *W* happens to be the definition of the Lambert-W function (Euler, 1783; Lambert, 1758). The function on the left-hand side of the equation is depicted on Figure 1:

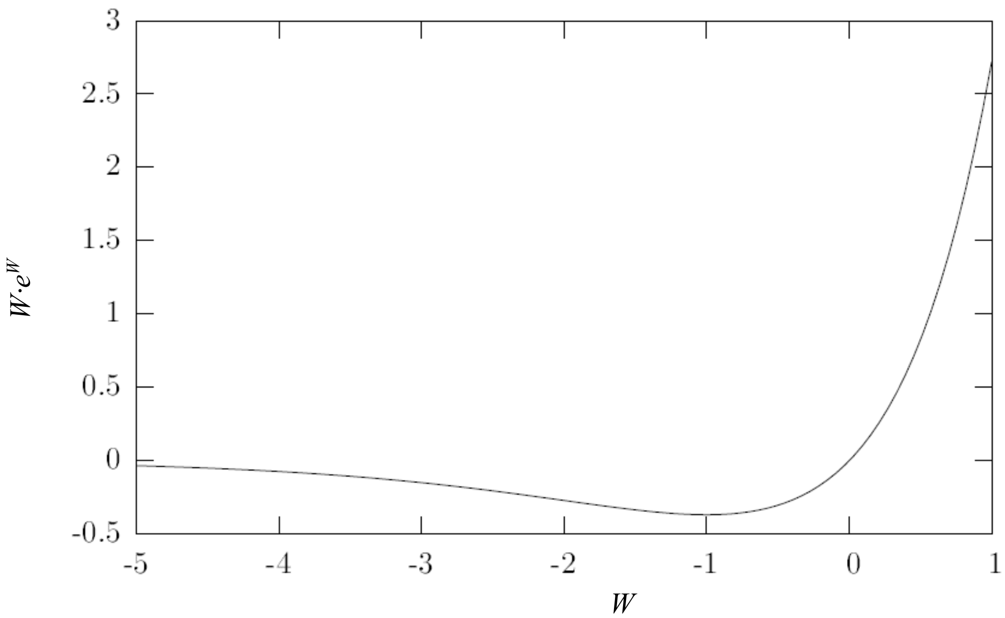

Figure 1. $W \cdot e^W$ as function of W.it should be noted from eq. (10.3.6) that the W and $W \cdot e^W$ is always negative in our case.

As we can deduct the value of W from the function in Figure 1, we can find the desired density parameters L_0 and a by substituting back:

$$a = -W/d_c \tag{10.2.7}$$

$$L_0 = \frac{L_m}{e^{-a \cdot d_c}} = L_m \cdot e^{a \cdot d_c} \tag{10.2.8}$$

1.1 Numeric solution to *W*

To solve the equation shown in Figure 1, we start by dividing the function into monotonic intervals by finding the derivative:

$$\frac{dWe^W}{dW} = e^W + We^W \tag{10.2.9}$$

The equation:

$$e^W + We^W = 0 \tag{10.2.10}$$

has one solution, $W = -1$. The expression $e^W(1+W)$ is decreasing below -1 and increasing above -1. Thus, $W = -1$ is a global minimum.

Since $lim_{W\rightarrow-\infty} We^W = 0$, we get a single solution for W when $-L_m \frac{d_c}{l_r}$ is exactly at the bottom point ($-1e^{-1}$), two when it is above (but not greater than 1), and none when it is below. The latter situation corresponds to a case, where there is insufficient root, l_r, to satisfy the minimal root density, L_m, within the total root zone depth, d_c.

Both solutions, if present, are valid, but represent different distributions.

- The solution for $W < -1$ represents a large a-parameter. From eq. (10.2.8) we see this also means that L_0 is large. Thus, the solution corresponds to a root zone with a high density near the top that decreases rapidly to L_m at the bottom of the root zone, and continues to decrease, so only a small contribution to the total root length will be present from below the root zone.
- The solution for $W > -1$ (and thus small values of a and L_0) corresponds to a low root density near the top that decreases slowly, and thus gives a larger contribution to the total root length from below the root zone.

As the total root length increases, pressing W towards 0 or -∞, the difference between the solutions grows. When there are just enough roots to satisfy the constraints at W = -1, the two solutions converge into one. As we prefer that the roots stay mostly within the root zone, we choose the solution for $W < -1$. We can thus find W numerically using Newton's method and an initial guess of -2.

1.2 Limited growth

The distribution in eq. (10.2.1) implies a gradual decrease of roots going towards, but never reaching zero. There are two problems with this. The first one is empirical. In some soils it does not match observations. Instead of a gradual decrease, there is a sharp decrease at a specific depth, as the roots are unable to penetrate further down. The other one is practical, as too large a root zone makes computation impractical.

The first problem is solved by dividing the root depth into a crop specific and soil independent potential root depth, $d_{c,pot}$, and a soil specific and crop independent maximum root depth, d_s. The actual root depth, d_a, is then the shallowest of these two.

$$d_a = min\left(d_{c,pot}, d_s\right) \tag{10.2.11}$$

We now create a modified root density function, L_z^*, by defining it to zero below d_a, and scaling L_z to preserve mass balance above d_a.

$$L_z^* = \begin{cases} k^* \cdot L_z & if\ z \leq d_a \\ 0 & if\ z > d_a \end{cases} \tag{10.2.12}$$

where

$$k^* = \frac{L_r}{\int_0^{d_a} L_z dz} \tag{10.2.13}$$

thus, solving both problems.

2 Row crops

We can describe a row crop with a two-dimensional model by assuming that the plants are densely packed in the row. Our second dimension, x, is horizontal, orthogonal to the row. The root density at a specific point can be denoted $L_{z,x}$, where we choose origo so $L_{0,0}$ is the root density in the middle of the row, see Figure 2.

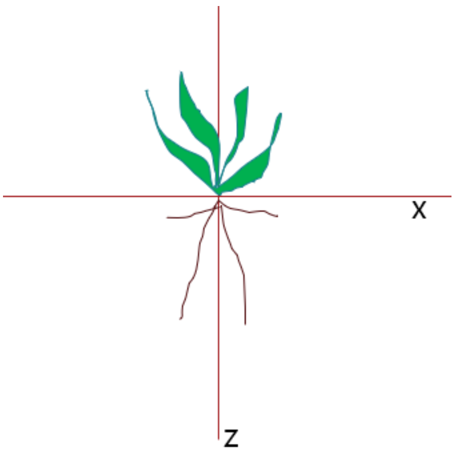

Figure 2. A row crop placed in (z,x) = (0,0). Other crops in the row are "behind" the plant. Other rows would be placed along the x-axis.

We then define the following root distribution:

$$L_{z,x} = L_{0,0} e^{-a_z z} e^{-a_x x} \tag{10.2.14}$$

where a_z and a_x control the density decrease along the z-axis and x-axis respectively (i.e. in the two dimensions).

2.1 Finding the parameters

To find the parameters a_z, a_x and $L_{0,0}$, we assume as before that the root depth and root mass is known, and now additionally that the radius of the horizontal extension of the roots, w_c, is known. We define the total root zone depth, d_c, to be the depth right below the root ($x = 0$) where the root density is L_m. As $x = 0$ is the place where eq. (10.2.14) predicts the highest density, the average root density at the depth will be well below L_m. Similarly, we define the radius w_c as the horizontal distance from the row, where the root density at the surface ($z=0$) equals L_m:

$$L_m = L_{d_c,0} = L_{0,w_c} \tag{10.2.15}$$

In the root module, $w_c = d_c \cdot MaxWidth/MaxPen$. *MaxWidth* and *MaxPen* are both parameters under *RootSystem,* describing the maximum width and penetration of the root system, respectively. If *MaxWidth* is not defined, it is set equal to *MaxPen* and $w_c = d_c$, e.g. the horizontal and vertical extent of the root zone is identical.

The total root length on one side of the row (l_R), which we assume is known from our crop model, is the integral of the root density over the half plane, as shown in eq. (10.2.16):

$$\begin{aligned} l_R &= \int_0^\infty \int_0^\infty L_{z,x} dz\, dx \\ &= \int_0^\infty \int_0^\infty L_{0,0} e^{-a_z z} e^{-a_x x} dz\, dx \\ &= \frac{L_{0,0}}{a_x a_z} \end{aligned} \tag{10.2.16}$$

$L_{0,0}$ in eq. (10.2.16) can be isolated and the resulting expression can be introduced in eq. (10.2.14):

$$L_{z,x} = l_R\, a_z a_x e^{-a_z z} e^{-a_x x} \tag{10.2.17}$$

Now, L_m in eq. (10.2.15) can be rewritten as:

$$L_m = l_R a_z a_x e^{-a_z d_c} \tag{10.2.18}$$

$$L_m = l_R a_z a_x e^{-a_x w_c} \tag{10.2.19}$$

Thus, $e^{-a_z d_c} = e^{-a_x w_c}$ or

$$a_x = \frac{d_c}{w_c} a_z \tag{10.2.20}$$

By inserting eq. (10.2.20) in (10.2. 19), L_m is expressed as:

$$L_m = l_R a_z \frac{d_c}{w_c} a_z e^{-a_z d} \tag{10.2.21}$$

We now introduce the variable Q:

$$Q = a_z d_c \tag{10.2.22}$$

and isolate the known values on the right side. This results in eq. (10.2.23):

$$Q^2 e^Q = L_m \frac{d_c \cdot w_c}{l_R} \tag{10.2.23}$$

The left-hand side expression of eq. (10.2.23) is illustrated in Figure 3.

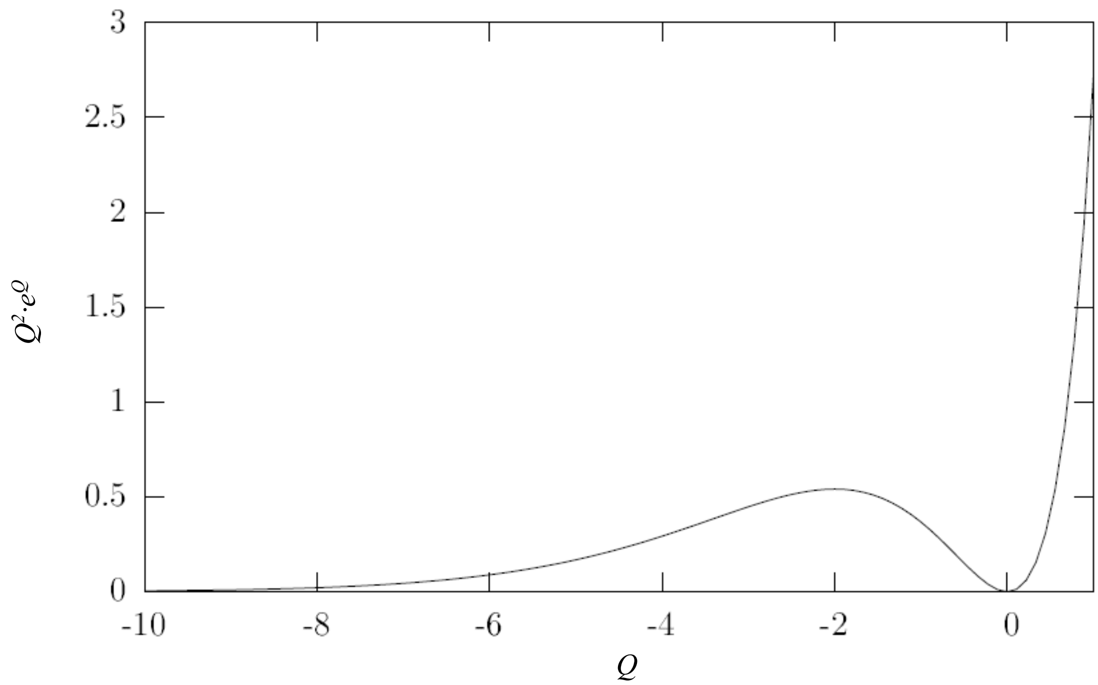

Figure 3. $Q^2 \cdot e^Q$ as function of Q.it should be noted from eq. (10.3.22) that the Q is always negative in our case.

2.2 Numeric solution to *Q*

First, the function is divided into monotonic intervals by finding the derivative:

$$\frac{d(Q^2 e^Q)}{dQ} = 2Qe^Q + Q^2 e^Q \tag{10.2.24}$$

The equation

$$2Qe^Q + Q^2 e^Q = 0 \tag{10.2.25}$$

has two solutions, $Q = 0$ and $Q = -2$. The expression $Q^2 e^Q$ is increasing below -2, decreasing between -2 and 0, and increasing above 0. Thus, $Q = 0$ is a local (and in this case also global) minimum and $Q = -2$ is a local maximum. We are now interested in positive values for Q, as they correspond to negative values for a_z and the simplification in eq. (10.2.16) is only valid if $a_z > 0$.

Since $lim_{Q\text{->}-\infty} Q^2 e^Q = 0$, there is a single negative solution when $L_m \frac{d_c \cdot w_c}{l_R}$ is exactly at the top-point ($2^2 e^{-2}$), two when it is smaller (it is never negative), and none when it is larger. The latter situation corresponds to the case where there is insufficient root l_R to satisfy the minimal root density L_m, within the given root zone $d_c \cdot w_c$.

Both negative solutions are valid, but represent different distributions:

- The solution for $Q < -2$ represents a large a_z (and thus also a_x) parameter. From eq. (10.2.16) it can be seen that this also means that $L_{0,0}$ is large. Thus, the solution corresponds to a root zone with a high density near the center that decreases rapidly to L_m near the edge og the root zone, and continues to decrease, so there is only a small contribution to the total root length from outside the root zone.
- The solution for $Q > -2$ (and thus small values of a_z, a_x, and $L_{0,0}$) corresponds to a low root density near the center that decreases slowly, and thus results in a larger contribution to the total root length from outside the root zone.

As the total root length increases, pressing Q towards 0 or -∞, the difference between the solutions grows. When there are just enough roots to satisfy the constraints at $Q = -2$, the two solutions converge into one. As the roots should preferably stay within the root zone, we choose the solution for $Q \leq -2$. We can find Q numerically, using Newton's method and an initial guess of -3. From that, a_z can be calculated from eq. (10.2.22), a_x from (10.2.20) and $L_{0,0}$ from eq. (10.2.16).

2.3 Multiple rows

If the rows are close enough, the root systems will overlap, as shown in Figure 4.

If R is the distance between rows, and we assume an infinite number of identical rows, this can be expressed by the equation:

$$L^*_{z,x} = \begin{cases} \sum_{i=0}^{\infty} \left(L_{z,x+iR} + L_{z,R+iR-x}\right) & if\ x < R/2 \\ 0 & if\ x \geq R/2 \end{cases} \tag{10.2.26}$$

Using eq. (10.2.14) and the rules for geometric series, the first case in eq. (10.2.26) can be re-written as

$$\begin{aligned} &\sum_{i=0}^{\infty} \left(L_{z,x+iR} + L_{z,R+iR-x}\right) \\ &= L_{0,0} e^{-a_z z} \sum_{i=0}^{\infty} \left(e^{-a_x(x+iR)} + e^{-a_x(R+iR-x)}\right) \\ &= L_{0,0} e^{-a_z z} \left(e^{-a_x x} \sum_{i=0}^{\infty} e^{-a_x iR} + e^{-a_x(R-x)} \sum_{i=0}^{\infty} e^{-a_x iR}\right) \\ &= L_{0,0} e^{-a_z z} \left(e^{-a_x x} + e^{-a_x(R-x)}\right) \sum_{i=0}^{\infty} e^{-a_x iR} \\ &= L_{0,0} e^{-a_z z} \left(e^{-a_x x} + e^{-a_x(R-x)}\right) \sum_{i=0}^{\infty} \left(\left(\frac{1}{e}\right)^{a_x R}\right)^i \\ &\frac{= L_{0,0} e^{-a_z z} \left(e^{-a_x x} + e^{-a_x(R-x)}\right)}{1 - \left(\frac{1}{e}\right)^{a_x R}} \end{aligned} \tag{10.2.27}$$

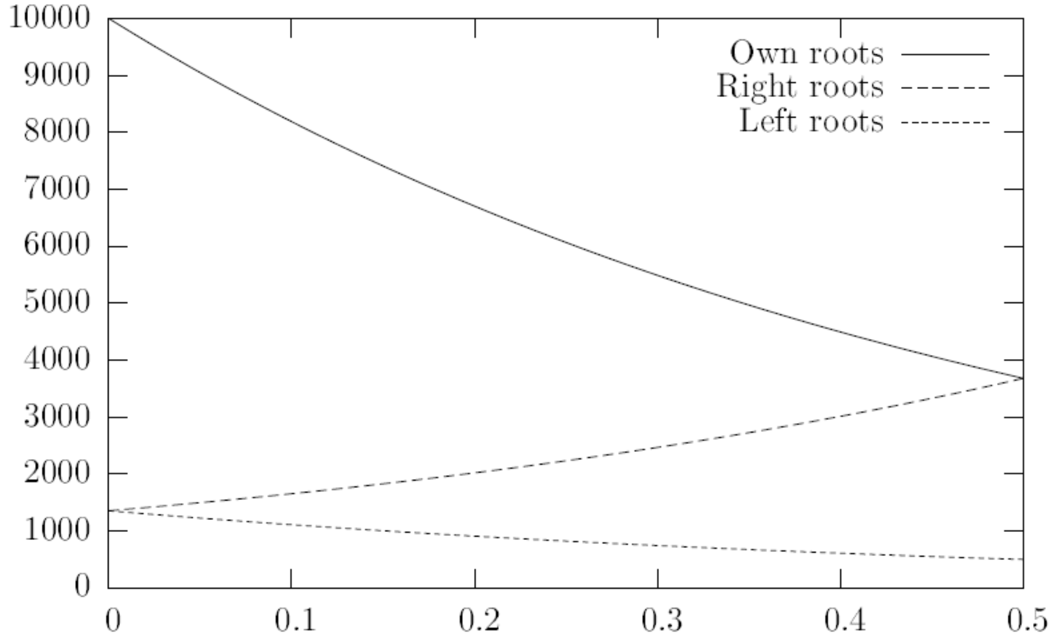

Figure 4. The x-axis represents the relative distance from a row to the midpoint between that row and the row to its right. The y-axis is the root density for roots originating in a specific row. The top line represents the roots from the row itself. The next line represents the roots from the row to the right. The last line represents the roots from the row to the left. In theory, all the rows on the field will contribute some roots to the interval. The root density in the interval will be the sum of all the individual contributions.

2.4 Mapping between the models

We would like to retain our original distribution when ignoring the x-dimension. We could not do that when looking only at the root system for a single row, as it is indefinitely wide and thus has an average density of zero. However, if we look at the roots of a single row, we get:

$$L_z = \frac{2\int_0^\infty L_{z,x}\,dx}{R} \tag{10.2.28}$$

We multiply by two, as we assume that two sides of the rows are identical. By integrating to ∞ rather than just to $R/2$, we do include roots from outside the row. However, because the system has an infinite number of identical rows, the amount of roots from the crop outside its own row is exactly the same as the amount of roots for other rows inside the row we are examining.

Inserting eq. (10.2.14) and (10.2.1) in (10.2.28), results in the following equation:

$$\begin{aligned} L_0 e^{-az} &= \frac{2}{R}\int_0^\infty L_{0,0} e^{-a_z z} e^{-a_x x} dx \\ &= \frac{2L_{0,0}e^{-a_z z}}{R}\int_0^\infty e^{-a_x x} dx \\ &= \frac{2L_{0,0}e^{-a_z z}}{R}\cdot\left(\frac{0-1}{-a_x}\right) \\ &= \frac{2L_{0,0}e^{-a_z z}}{R\cdot a_x} \end{aligned} \tag{10.2.29}$$

The relationship between the parameters in 1 and 2 dimensions are therefore as follows:

$$a_z = a \tag{10.2.30}$$

$$L_{0,0} = \frac{1}{2} a_x R L_0 \tag{10.2.31}$$

$$L_0 = \frac{2L_{0,0}}{a_x R} \tag{10.2.32}$$

These equations are used to switch between the one- and two-dimensional descriptions.

3 Parameters

Table 10.2.1. List of symbols

Symbol	Unit	Description
a	m^{-1}	Root density distribution parameter (vertical, 1D)
a_z	m^{-1}	Vertical root density distribution parameter
a_x	m^{-1}	Horizontal root density distribution parameter
d_a	m	Actual root depth, limited by soil and crop constraints
d_c	m	Root depth
$d_{c,pot}$	m	Crop potential root depth
d_s	m	Soil maximum root depth
k^*		Soil root limit factor
l_r	$m\ m^{-2}$	Total root length per area
l_R	$m\ m^{-1}$	Total root length per length of row on one side
L_0	$m\ m^{-3}$	Average root density at soil surface
$L_{0,0}$	$m\ m^{-3}$	Root density in row at the soil surface
L_m	$m\ m^{-3}$	Minimal root density
L_z	$m\ m^{-3}$	Root density at soil depth z
L_z^*	$m\ m^{-3}$	Soil limited root density at soil depth z
$L_{z,x}$	$m\ m^{-3}$	Root density at soil depth z and distance x from row
$L^*_{z,x}$	$m\ m^{-3}$	Root density from multiple rows
M_r	$m\ m^{-2}$	Total root dry matter
Q		Substitution variable
R	m	Distance between rows
S_r	$m\ kg^{-1}$	Specific root length
W		Lambert-W-function
w_c	m	Extension of roots in the horizontal direction, radius
x	m	Horizontal distance from row
z	m	Soil depth

Table 10.2.2. Related Parameter names in Daisy.

Name and explanation		**Model (in Daisy)**	**Parameter name (Daisy reference manual)**	**Default**	**Default unit**
$d_{c,pot}$	Maximum penetration depth based on crop	RootSystem	*MaxPen*	Default = 100	[cm]
MaxWidth	Maximum horizontal distance of roots from plant.	RootSystem	*MaxWidth*	Optional parameter, default = *MaxPen*	[cm]
d_s	The half-saturation constant for bio-incorporation, Eq. (9.1).	Soil	*MaxRootingDepth*	Optional parameter	[cm]
S_r	Specific root length	Rootdens	*SpRtLength*	Default = 100	[m g^{-1}]
L_m	Root density at (potential) penetration depht	GPD1 GPD2	*DensRtTip*	Default = 0.1	[cm cm^{-3}]
	Ignore cells with less than this root density.	GPD1 GPD2	*DensIgnore*	Optional parameter, by default the same as *DensRtTip*	[cm cm^{-3}]
	Position of plant row on x-axis	action, sow_base	*row_position*	parameter (default 0), equal to no rows. But if row_width is > 0, e.g. there are rows, the 0 will be 0 on the x-axis.	[cm]
R	Distance between rows of crops	action, sow_base	*row_width*	default = 0, indicating equal spreading of seed over the area (no rows)	[cm]

Original text from	D3_2_safir	
Updated by	date	For Daisy version
Styczen, M	2024-11-06	6.47

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